

Expected Behavior of Quantum Thermodynamic Machines with Prior Information

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We estimate the expected behavior of a quantum model of heat engine when we have incomplete information about external macroscopic parameters, like magnetic field controlling the intrinsic energy scales of the working medium. We explicitly derive the prior probability distribution for these unknown parameters, a_i , ($i = 1, 2$). Based on a few simple assumptions, the prior is found to be of the form $\Pi(a_i) \propto 1/a_i$. By calculating the expected values of various physical quantities related to this engine, we find that the expected behavior of the quantum model exhibits thermodynamic-like features. This leads us to a surprising proposal that incomplete information quantified as appropriate prior distribution can lead us to expect classical thermodynamic behavior in quantum models.

PACS numbers: 05.70.-a, 03.65.-w, 05.70.Ln, 02.50.Cw

I. INTRODUCTION

In a recent paper [1], a connection was observed between the notion of prior probabilities [2, 3] and the optimal performance of certain models of quantum heat engines [4]. As a central feature of Bayesian inference, these probabilities quantify the uncertainty or equivalently, the lack of complete knowledge about some parameters of the problem. It was found [1] that a power-law choice of the prior, yields the expected optimal performance of the quantum engine at certain well-known thermal efficiencies such as Curzon-Ahlborn (CA) efficiency [5]. This observation is remarkable from at least two viewpoints: i) the CA efficiency is well-known to be a finite-time thermodynamic result, where it arises within endoreversible thermodynamic models as efficiency at maximum power. But the cycle considered in Ref. [1] runs infinitely slow; ii) the derivation of this result based on Bayesian approach suggests a very different origin of these efficiencies by connecting them with the notion of incomplete information.

Let us clarify about the source of the assumed incomplete information. Usually, incomplete information about the state of a quantum system is meant to refer to an uncertain preparation procedure, the state being defined by a mixed state density matrix $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. This ensemble is characterised by a hamiltonian $H(a)$ of the system, a function of externally controllable parameters (collectively denoted by a), such as magnetic field, coupling strength with environment/subsystems and so on. One may say that the state or the ensemble is characterised by the choice of these external parameters. In the present discussion, we assume an ignorance about the exact values of these parameters. Our focus is on the following issue: can someone who is ignorant of the exact values of certain control parameters for the system, make some reasonable estimates about its behavior? The particular physical problem that we choose is the performance of heat engines. In particular, we ask how should one quantify one's ignorance about the configuration of a quantum heat engine and what are the estimates for its expected performance?

Now we may possess some prior information about a parameter which is uncertain in the above sense, such as the

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possible range of values or, other inputs from the physics of the problem. Based on that we assign probabilities for the likely values of this parameter. However, these probabilities are to be interpreted in the sense of rational degree of belief [2]. Thus essentially we interpret our ignorance about parameters of the system from a Bayesian perspective. Formally, one may say that we are considering an ensemble of subensembles, where each subensemble is characterised by a hamiltonian $H(a)$, and the possible values of a over different subsensembles are described by a prior distribution. We may refer to the ensemble so considered as the *prior ensemble*, but the ensemble is regarded here as a theoretical construct only. Our aim is to quantify the prior information about the uncertain parameters as a prior distribution and to use this prior to make estimates about the behavior of the system.

The purpose of this paper is to derive and justify the choice of the prior from more basic underlying assumptions and to further clarify the subjective approach. Our quantum model of a heat engine consists of a pair of weakly interacting two-level systems, which is closely analogous to the model of [1, 4]. There are two reservoirs characterised by temperatures T_1 and T_2 . Additionally, we have two energy scales, denoted by $a_i (i = 1, 2)$ referring to the level spacing of a two-level system. These spacings are controllable externally e.g. via magnetic field, but are here assumed to be uncertain. It is these energy scales for which we assign a prior. The form of the prior we derive for parameter a_i , is the $1/a_i$ law. We show the consequences of using such a prior for the expected values of physical quantities such as efficiency, work and so on. We observe that the assignment of the prior is such that it leads us to *expect* a thermodynamic like behavior in these quantum models.

The paper is organised as follows. In Section II, we present the model for heat engine and summarise its main features. In Section III, we assume incomplete information about internal energy scales of the working medium and so derive the prior for them based on some simple assumptions. In Section IV, we apply the prior so derived to estimate thermal efficiency of the engine, final temperature, work extracted and so on. In Section V, we highlight a specific asymptotic limit in which the expected behavior becomes especially simple. Section VI, addresses a special case by including a constraint. The final Section VII is devoted to conclusions and future outlook.

II. QUANTUM MODEL FOR WORK EXTRACTION

Consider a pair of two-level systems labeled R and S, with hamiltonians H_R and H_S having energy eigenvalues $(0, a_1)$ and $(0, a_2)$, respectively. The hamiltonian of the composite system is given by $H = H_R \otimes I + I \otimes H_S$. The initial state is $\rho_{\text{ini}} = \rho_R \otimes \rho_S$, where ρ_R and ρ_S are thermal states corresponding to temperatures T_1 and T_2 ($< T_1$), respectively. Let (r_1, r_2) and (s_1, s_2) be the occupation probabilities of each system, where

$$r_1 = \frac{1}{(1 + e^{-a_1/T_1})}, \quad s_1 = \frac{1}{(1 + e^{-a_2/T_2})}, \quad (1)$$

with $r_2 = (1 - r_1)$ and $s_2 = (1 - s_1)$. We have set Boltzmann's constant $k_B = 1$. The initial mean energy of each system is

$$E_{\text{ini}}^{(i)} = \frac{a_i}{(1 + e^{a_i/T_i})}, \quad (2)$$

where $i = 1, 2$ denote system R and S respectively. Within the approach based on quantum thermodynamics [6–8], the process of maximum work extraction is identified as a quantum unitary process on the thermally isolated

composite system. It preserves not just the magnitude of von Neumann entropy of the composite system, but also all eigenvalues of its density matrix. It has been shown in these works that for $a_1 > a_2$, such a process minimises the final energy if the final state is given by $\rho_{\text{fin}} = \rho_S \otimes \rho_R$. Effectively, it means that in the final state the two systems *swap* between themselves their initial probability distributions. The final energy of each system at the end of work extracting transformation is

$$E_{fin}^{(i)} = \frac{a_i}{(1 + e^{a_j/T_j})}. \quad (3)$$

where $i \neq j$. The average work per cycle defined as $W \equiv \text{Tr}[(\rho_{\text{ini}} - \rho_{\text{fin}})H] = E_{\text{ini}} - E_{\text{fin}}$, is given by

$$W(a_1, a_2) = (a_1 - a_2) \left[\frac{1}{(1 + e^{a_1/T_1})} - \frac{1}{(1 + e^{a_2/T_2})} \right]. \quad (4)$$

To complete the cycle, the two systems are brought again in thermal contact with their respective reservoirs. In this cycle, the heat extracted from the hot reservoir is

$$Q_1 = a_1 \left[\frac{1}{(1 + e^{a_1/T_1})} - \frac{1}{(1 + e^{a_2/T_2})} \right]. \quad (5)$$

The efficiency of this engine $\eta = W/Q_1$ is

$$\eta = 1 - \frac{a_2}{a_1}. \quad (6)$$

Note that for $a_2 = a_1$, $W = 0$, $Q_1 > 0$ and $\eta = 0$; for $a_2 = a_1(T_2/T_1)$, we have the limiting values of $W = 0$ and $Q_1 = 0$ and $\eta = 1 - (T_2/T_1)$. The operation of the machine as a heat engine defined as $W \geq 0$ and $Q_1 \geq 0$, is satisfied if

$$a_1(T_2/T_1) \leq a_2 \leq a_1. \quad (7)$$

III. BAYESIAN APPROACH

Now consider a situation in which the temperatures of the reservoirs are given a priori such that $T_1 > T_2$, but about the parameters a_1 and a_2 we only know that,

- a_1 and a_2 represent the same physical quantity, which is level spacing for system R and S respectively, and so they can only take positive real values.
- If the set-up of R+S has to work as an engine, then criterion in Eq. (7) must hold, whereby if one parameter is specified, then it redefines the range of the other parameter.

Apart from the above conditions, we assume to have no information about a_1 and a_2 . The question we address in the following is: What can we then infer about the expected behaviour of physical quantities for this heat engine (such as work per cycle, efficiency and so on) ? We shall follow a subjective approach to probability to address this question. This implies that an uncertain parameter is assigned a prior distribution, which quantifies our preliminary expectation about the parameter to take a certain value. We denote the prior distribution function for our problem

by $\Pi(a_1, a_2)$. The prior should be assigned by taking into account any prior knowledge or information we possess about the parameters. For example, if a_1 is specified, then the prior distribution for a_2 , $\pi(a_2|a_1)$ is conditioned on the specified value of a_1 , and is defined in the range $[a_1\theta, a_1]$, where $\theta = T_2/T_1$, because we know the set-up works like an engine if we implement Eq. (7).

The expected value of any physical quantity X which may be function of a_1 and a_2 , is defined as follows:

$$\overline{X} = \int \int X \Pi(a_1, a_2) da_1 da_2. \quad (8)$$

A. Assignment of the prior

In Bayesian probability theory, the assignment of a unique prior is a central issue. It should quantify not only the prior knowledge about the parameter in the particular context, but should meet a consistency criterion according to which different observers in possession of equivalent information should assign similar priors. Suppose, one knows only the range within which the parameter takes its values, then intuition suggests that a uniform distribution may reflect the state of our knowledge. But such a choice is not invariant under reparameterizations. Our aim in the following is to motivate the assignment of prior distributions for level spacings a_1 and a_2 , when the physical problem at hand is a heat engine as described in Section (II).

It seems convenient to speak in terms of the two observers A and B, who wish to assign priors for a_1 and a_2 . The assignment is based on the following assumptions:

- (a) Same functional form of the prior is assigned to a_1 and a_2 in the *initial state*, denoted by $\Pi(a_1)$ and $\Pi(a_2)$ respectively. Further, we assume that the prior can be expressed as $\Pi(a_i) \propto df(a_i)/da_i$, using a continuous differentiable function $f(a_i)$ ($i = 1, 2$).
- (b) The conditional prior distributions $p(a_j|a_i)$, implying distribution for a_j given a value of a_i , or $p(a_i|a_j)$ for the converse case, have the same functional form as above, and we assume that $p(a_j|a_i) \propto df(a_j)/da_j$, where the dependence on a_i may be present in the normalisation factor. Similarly, we assume $p(a_i|a_j) \propto df(a_i)/da_i$.

Assumption (a) is reasonable since both a_1 and a_2 represent the same physical quantity, and the state of knowledge of A and B about them is the same, which is the fact that their values *in the initial state* lie in a preassigned range $[a_{\min}, a_{\max}]$. For simplicity and symmetry, we take this range to be identical for a_1 and a_2 . Presumably, this range depends on the experimental setup, and we assume similar apparatus for controlling the level spacings of systems R and S.

Now if one of the parameters is specified to an observer, say a_1 to A, then A knows that the machine works as an engine only if the range of a_2 is $[a_1\theta, a_1]$, where a_1 here represents some fixed value. Even so, assumption (b) states that A must assign the same functional form to the prior for a_2 as it used for assigning to a_1 . Although this does not represent the general case, we assume this for simplicity and in the following analyse the consequences of these assumptions.

Thus from (a), we have

$$\Pi(a_i) = \frac{1}{M} \frac{df(a_i)}{da_i}, \quad (9)$$

where the normalization constant is determined as

$$\begin{aligned} M &= \int_{a_{\min}}^{a_{\max}} \frac{df(a_i)}{da_i} da_i \\ &= f(a_{\max}) - f(a_{\min}). \end{aligned} \quad (10)$$

Following assumption (b), the conditional probability distributions are given by

$$\Pi(a_2|a_1) = \frac{1}{N_1} \frac{df(a_2)}{da_2}, \quad (11)$$

and

$$\Pi(a_1|a_2) = \frac{1}{N_2} \frac{df(a_1)}{da_1}, \quad (12)$$

where

$$\begin{aligned} N_1 &= \int_{a_1\theta}^{a_1} \frac{df(a_2)}{da_2} da_2 \\ &= f(a_1) - f(a_1\theta), \end{aligned} \quad (13)$$

and

$$\begin{aligned} N_2 &= \int_{a_2}^{a_2/\theta} \frac{df(a_1)}{da_1} da_1 \\ &= f(a_2/\theta) - f(a_2), \end{aligned} \quad (14)$$

are the respective normalization constants. Now using the product law of probabilities, the joint prior $\Pi(a_1, a_2)$ as expressed by observer A

$$\begin{aligned} \Pi(a_1, a_2) &= \Pi(a_2|a_1) \cdot \Pi(a_1) \\ &= \frac{df(a_2)/da_2}{[f(a_1) - f(a_1\theta)]} \cdot \frac{df(a_1)/da_1}{[f(a_{\max}) - f(a_{\min})]}, \end{aligned} \quad (15)$$

or equivalently in terms of B,

$$\begin{aligned} \Pi(a_1, a_2) &= \Pi(a_1|a_2) \cdot \Pi(a_2) \\ &= \frac{df(a_1)/da_1}{[f(a_2/\theta) - f(a_2)]} \cdot \frac{df(a_2)/da_2}{[f(a_{\max}) - f(a_{\min})]}. \end{aligned} \quad (16)$$

As shown in the Fig. 1, each observer assigns different ranges of values to a_1 and a_2 . Each can use its own joint prior to make estimates about any quantity. Now since either approach, that of A or B, is equivalent, consistency would require that they make similar estimates as to a given quantity. Also it is reasonable to assume that for some given pair of values (a_1, a_2) , which are in the allowed range of each observer, each of them should assign same probability for choosing a_1 within the small interval da_1 around the given value a_1 , as well as of choosing a_2 within the small interval da_2 , around the given value a_2 . Clearly, for such a pair, both a_1 and a_2 have to lie in the interval $[a_{\min}, a_{\max}]$. One such pair of values, which is definitely common to both A and B would be (a_1, a_1) i.e. when $a_2 = a_1$. Thus equating the probabilities assigned by A and B for this case, we obtain from Eqs. (15) and (16),

$$f(a_1) - f(a_1\theta) = f(a_1/\theta) - f(a_1), \quad (17)$$

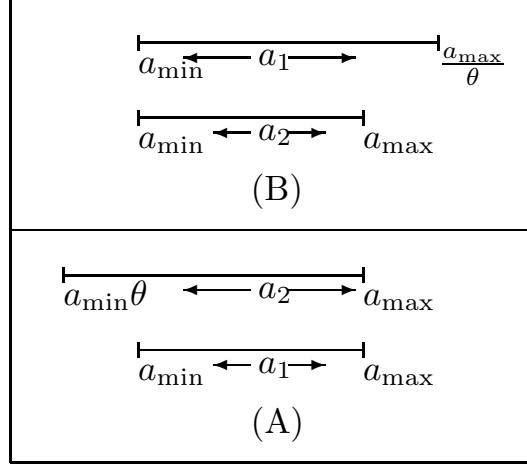


FIG. 1: Observer A (B) assigns the same range of values for a_1 (a_2) in the initial state of spin R (S). But the range assigned for the other parameter a_2 (a_1), conditional on the operation as an engine, is different. This fact manifests such that A and B in general arrive at different estimates for physical quantities.

which can be rewritten as

$$2f(a_1) = f(a_1\theta) + f(a_1/\theta). \quad (18)$$

This functional equation has a unique solution, $f(x) = \lambda \ln x$, upto an additive constant. Thus we find the explicit form of prior (Eq. (9)) as

$$\Pi(a_i) = \frac{1}{\ln\left(\frac{a_{\max}}{a_{\min}}\right)} \frac{1}{a_i}, \quad (19)$$

which is functionally the same as Jeffreys' choice [2], also employed in a previous study [1]. The joint prior is then given by

$$\Pi(a_1, a_2) = \frac{1}{\ln\left(\frac{1}{\theta}\right) \ln\left(\frac{a_{\max}}{a_{\min}}\right)} \frac{1}{(a_1 a_2)}. \quad (20)$$

The joint prior derived above is relevant only with regard to the final states of R and S. As discussed in the previous section, the initial state of system R (S) depends only on parameter a_1 (a_2). This fact will be used in the following in calculations on the expected values of quantities.

IV. EXPECTED VALUES OF PHYSICAL QUANTITIES

In this section, we use the priors assigned above, to find expected values for various physical quantities related to the engine. For this purpose, we employ the definition in Eq. (8). These expected values reflect the estimates by an observer who is assigning the priors. As observed in Eqs. (15) and (16), there are two methods of writing the joint prior. So in principle, there are two ways to calculate the expected value of some quantity and its value will depend, in general, on the method used.

A. Internal energy

We calculate the expected values of internal energies for systems R and S. These values can then be used to find the expected work per cycle, heat exchanged and so on.

(i) **Initial state:** For a given a_i , the internal energy $E_{ini}^{(i)}$ is given by (2). The expected initial energy is defined as

$$\overline{E}_{ini}^{(i)} = \int_{a_{\min}}^{a_{\max}} E_{ini}^{(i)} \Pi(a_i) da_i, \quad (21)$$

where $i = 1, 2$. Note that $E_{ini}^{(i)}$ depends only on a_i , so we need to average over the prior for a_i only. Thus we obtain

$$\overline{E}_{ini}^{(i)} = \left[\ln \left(\frac{a_{\max}}{a_{\min}} \right) \right]^{-1} \int_{a_{\min}}^{a_{\max}} \frac{da_i}{(1 + e^{a_i/T_i})}, \quad (22)$$

or explicitly

$$\overline{E}_{ini}^{(i)} = \left[\ln \left(\frac{a_{\max}}{a_{\min}} \right) \right]^{-1} \left[(a_{\max} - a_{\min}) + T_i \ln \left(\frac{1 + e^{a_{\min}/T_i}}{1 + e^{a_{\max}/T_i}} \right) \right]. \quad (23)$$

(ii) **Final state :** In this case, the internal energy of R as well as S, is function of both a_1 and a_2 (see (3)) and so the expected values are obtained by averaging over the joint prior, $\Pi(a_1, a_2)$. For instance, the expected final energy of system S (denoted by superscript (2)) as calculated by A,

$$\begin{aligned} \overline{E}_{fin}^{(2)}(A) &= K \int_{a_{\min}}^{a_{\max}} \frac{1}{(1 + e^{a_1/T_1})a_1} da_1 \int_{a_1\theta}^{a_1} da_2, \\ &= K (1 - \theta) \left[(a_{\max} - a_{\min}) + T_1 \ln \left(\frac{1 + e^{a_{\min}/T_1}}{1 + e^{a_{\max}/T_1}} \right) \right]. \end{aligned} \quad (24)$$

Similarly if calculated by B,

$$\overline{E}_{fin}^{(2)}(B) = K \int_{a_{\min}}^{a_{\max}} da_2 \int_{a_2}^{a_2/\theta} \frac{da_1}{(1 + e^{a_1/T_1})a_1}, \quad (25)$$

where $K = [\ln(1/\theta) \ln(a_{\max}/a_{\min})]^{-1}$. The latter integral cannot be completely solved. To simplify, we rewrite it as

$$\overline{E}_{fin}^{(2)}(B) = K \int_{a_{\min}}^{a_{\max}} da_2 \left[\int_{a_2}^{a_2/\theta} \frac{da_1}{(1 + e^{a_1/T_1})a_1} \cdot 1 \right]. \quad (26)$$

Considering the inner integral as the first function and unity as the second function and integrating by parts, leads to

$$\begin{aligned} \overline{E}_{fin}^{(2)}(B) &= K a_2 \int_{a_2}^{a_2/\theta} \frac{da_1}{(1 + e^{a_1/T_1})a_1} \Big|_{a_2=a_{\min}}^{a_2=a_{\max}} \\ &\quad - K \int_{a_{\min}}^{a_{\max}} a_2 \left[\frac{d}{da_2} \int_{a_2}^{a_2/\theta} \frac{da_1}{(1 + e^{a_1/T_1})a_1} \right] da_2. \end{aligned} \quad (27)$$

Here we use Leibniz integral rule

$$\frac{d}{dy} \int_{g(y)}^{h(y)} f(x) dx = \frac{dh(y)}{dy} f(h(y)) - \frac{dg(y)}{dy} f(g(y)), \quad (28)$$

to solve the second term of the Eq. (27) and to finally obtain

$$\begin{aligned} \overline{E}_{fin}^{(2)}(B) &= K a_2 \int_{a_2}^{a_2/\theta} \frac{da_1}{(1 + e^{a_1/T_1})a_1} \Big|_{a_2=a_{\min}}^{a_2=a_{\max}} \\ &\quad - K \left[T_2 \ln \left(\frac{1 + e^{a_{\min}/T_2}}{1 + e^{a_{\max}/T_2}} \right) - T_1 \ln \left(\frac{1 + e^{a_{\min}/T_1}}{1 + e^{a_{\max}/T_1}} \right) \right]. \end{aligned} \quad (29)$$

In general, the expected final energies of S, as given by Eqs. (24) and (29) according to A and B, respectively, are not equal. One would expect that if the state of knowledge of A and B is similar, then they should expect the same value for a given quantity. However, the difference in the values expected by A and B is not so surprising in light of the fact that different ranges for variables a_1 and a_2 are being employed by them, as shown in Fig. 1.

Similar feature is also observed in the expressions for expected energy of system R (superscript (1)), which we provide below for sake of completeness.

$$\overline{E}_{fin}^{(1)}(A) = \int \int E_{fin}^{(1)} \Pi(a_2|a_1) \Pi(a_1) da_2 da_1, \quad (30)$$

which implies

$$\overline{E}_{fin}^{(1)}(A) = K \int_{a_{min}}^{a_{max}} da_1 \int_{a_1 \theta}^{a_1} \frac{da_2}{(1 + e^{a_2/T_2}) a_2}. \quad (31)$$

It is interesting though to observe that the above integral is *identical* to the one in Eq. (25),

$$\overline{E}_{fin}^{(1)}(A) = \overline{E}_{fin}^{(2)}(B). \quad (32)$$

The second method however, yields

$$\begin{aligned} \overline{E}_{fin}^{(1)}(B) &= K \int_{a_{min}}^{a_{max}} \frac{da_2}{(1 + e^{a_2/T_2}) a_2} \int_{a_2}^{a_2/\theta} da_1, \\ &= K \left(\frac{1}{\theta} - 1 \right) \left[(a_{max} - a_{min}) + T_2 \ln \left(\frac{1 + e^{a_{min}/T_2}}{1 + e^{a_{max}/T_2}} \right) \right]. \end{aligned} \quad (33)$$

In the next section, we look at these expressions in a particular limit in which the expected values obtained by the two observers yield similar results, so that a meaningful analysis can be carried out in this limit.

V. ASYMPTOTIC LIMIT

As remarked above, observers A and B are supposed to arrive at similar conclusions. So they should arrive at similar estimates for physical quantities using their respective priors. This happens in the limit, when $a_{min} \ll T_2$ and $a_{max} \gg T_1$. In this limit, Eq. (23) is approximated as

$$\overline{E}_{ini}^{(i)} \approx \frac{\ln 2}{\ln(\frac{a_{max}}{a_{min}})} T_i. \quad (34)$$

The ratio (a_{max}/a_{min}) in the above may be large in magnitude, but is assumed to be finite.

Similarly, it is remarkable to note that in this limit, not only the expected energy of a system (R or S) calculated by either of the methods (A or B), is the same but also its value for system R or S is also equal. In particular, the first term of Eq. (29) can be shown to be negligible in this limit. Thus we have (omitting the observer index)

$$\overline{E}_{fin}^{(i)} \approx \frac{\ln 2}{\ln(\frac{a_{max}}{a_{min}})} \frac{(1 - \theta) T_1}{\ln(\frac{1}{\theta})}, \quad (35)$$

where $i = 1, 2$. Further insight into this may be obtained if we estimate the final temperatures of systems R and S after the work extraction process. Now if values of both a_1 and a_2 are specified, the temperatures (T'_i) of the two systems after work extraction, are given by [8]

$$T'_1 = T_2 \frac{a_1}{a_2}, \quad \text{and} \quad T'_2 = T_1 \frac{a_2}{a_1}. \quad (36)$$

In general, the two final temperatures are different from each other. Within the present framework, when we look at the expected values of the final temperatures as calculated by A or B, we find

$$\overline{T}'_1 = \overline{T}'_2 = T_1 \frac{(1 - \theta)}{\ln(1/\theta)}. \quad (37)$$

It is interesting to find that the assignment of the prior is such that the two systems are expected to finally arrive at a common temperature. Going back to Eqs. (34) and (35) for the energies, we see that they satisfy a simple relation $\overline{E}_{ini}^{(i)} \propto T_i$ and $\overline{E}_{fin}^{(i)} \propto \overline{T}'_i$. This is reminiscent of the thermodynamic behavior of a classical ideal gas.

Next, the heat exchanged between system i and the corresponding reservoir is given by $\overline{Q}_i = \overline{E}_{ini}^{(i)} - \overline{E}_{fin}^{(i)}$. $\overline{Q}_i > 0$ ($\overline{Q}_i < 0$) represents heat absorbed (released) by the system. Then the expressions for the heat exchanged with the reservoirs in the said limit are as follows:

$$\overline{Q}_1 \approx \frac{\ln 2}{\ln\left(\frac{a_{\max}}{a_{\min}}\right)} \left(1 + \frac{(1 - \theta)}{\ln \theta}\right) T_1, \quad (38)$$

and

$$\overline{Q}_2 \approx \frac{\ln 2}{\ln\left(\frac{a_{\max}}{a_{\min}}\right)} \left(1 + \frac{(1 - \theta)}{\theta \ln \theta}\right) T_2. \quad (39)$$

Now the expected work per cycle is defined as: $\overline{W} = \overline{Q}_1 + \overline{Q}_2$. Thus the efficiency may be defined as $\eta = 1 + \overline{Q}_2/\overline{Q}_1$. Explicitly, using Eqs. (38) and (39) we get

$$\eta = 1 + \frac{\theta \ln \theta + (1 - \theta)}{\ln \theta + (1 - \theta)}. \quad (40)$$

This is the efficiency at which the engine is expected to operate for a given θ . The above value is function only of the ratio of the reservoir temperatures and is plotted in Fig. 2. In the limit of small temperature differences,

$$\eta \approx \frac{(1 - \theta)}{3} + \frac{(1 - \theta)^2}{9} + O(1 - \theta)^3. \quad (41)$$

Before closing this section, we note that the constant of proportionality in Eqs. (34) and (35), which is $\ln 2 \cdot$

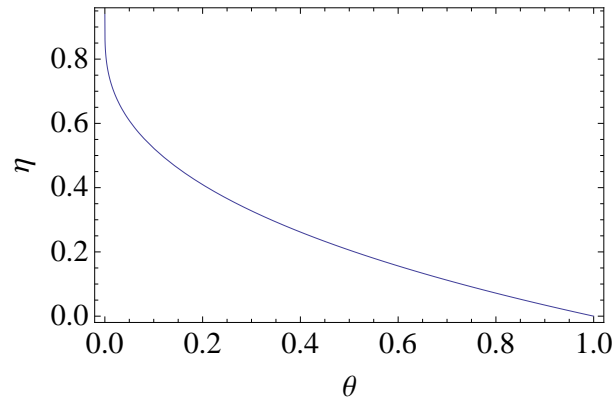


FIG. 2: The estimated efficiency of the engine based on expected values of heats exchanged with the reservoirs (Eqs. (38) and (39)). For near equilibrium, upto first order, the efficiency is 1/3 of Carnot value.

$(\ln(a_{\max}/a_{\min}))^{-1}$, can be related with heat capacity. The expected value of initial heat capacity of system i , defined as

$$\overline{C}_i = \int_{a_{\min}}^{a_{\max}} C_i \Pi(a_i) da_i, \quad (42)$$

where we know

$$C_i = \left(\frac{a_i}{T_i}\right)^2 \frac{e^{a_i/T_i}}{(1 + e^{a_i/T_i})^2}. \quad (43)$$

Upon solving, the expected heat capacity in the initial state of the system is given exactly by

$$\overline{C}_i = \left[\ln \left(\frac{a_{\max}}{a_{\min}} \right) \right]^{-1} \left[\frac{a_{\max} e^{a_{\max}/T_i}}{T_i(1 + e^{a_{\max}/T_i})} - \frac{a_{\min} e^{a_{\min}/T_i}}{T_i(1 + e^{a_{\min}/T_i})} + \ln \left(\frac{1 + e^{a_{\min}/T_i}}{1 + e^{a_{\max}/T_i}} \right) \right]. \quad (44)$$

Then in the asymptotic limit, the leading term yields

$$\overline{C}_i \equiv \overline{C} \approx \frac{\ln 2}{\ln \left(\frac{a_{\max}}{a_{\min}} \right)}. \quad (45)$$

This limiting value is independent of temperature of the system and thus indicates an analogy with a constant heat capacity thermodynamic system.

Closing this section, we note that the requirement of consistency between the results of A and B implies, in an asymptotic limit, that the behavior expected from minimal prior information is the one which shows simple thermodynamic features such as constant heat capacity and equality of subsystem temperatures upon maximum work extraction. In the next section, we revisit the above analysis, but in much simplified form by considering an additional constraint.

VI. ANALYSIS AT GIVEN THERMAL EFFICIENCY

Let us impose the additional constraint by fixing the value of engine efficiency η . Then there is essentially one energy scale say a_1 , to be specified in the model, because the other scale a_2 is determined from the ratio $a_2/a_1 = (1 - \eta)$. Let observer A treat a_1 as the uncertain parameter and denote the prior as $\pi(a_1)$. The second observer B chooses a_2 as the uncertain parameter and the corresponding prior as $\pi^*(a_2)$. In other words, instead of a_1 , B parametrises the uncertain quantity as a_1 times some given constant which in the present case is $(1 - \eta)$. The probabilities assigned by A and B for a given choice of a_1 and a_2 , must satisfy

$$\pi(a_1) da_1 = \pi^*(a_2) da_2. \quad (46)$$

On the other hand, one should expect the same functional form for the prior in both cases, which is saying essentially that both A and B are in an identical state of knowledge, implying $\pi \sim \pi^*$ [3]. Thus relation (46) is rewritten as

$$\pi(a_1) = (1 - \eta) \pi(a_1(1 - \eta)), \quad (47)$$

a functional equation whose solution is given by $\pi(x) \propto 1/x$. Thus with the additional constraint of a given efficiency, the prior can be assigned as the Jeffreys' prior.

Let us now illustrate the calculation for observer A. The appropriate normalised prior we have to consider is $\pi(a_1) = [\ln(a_{\max}/a_{\min})]^{-1} (1/a_1)$. From Eq. (4), the average work per cycle rewritten as function of a_1 and η is given by

$$W(a_1, \eta) = a_1 \eta \left[\frac{1}{(1 + e^{a_1/T_1})} - \frac{1}{(1 + e^{a_1(1-\eta)/T_2})} \right]. \quad (48)$$

The expected work estimated by A is defined as $\overline{W}(A) = \int W(a_1, \eta) \pi(a_1) da_1$. After calculation we have,

$$\overline{W}(A) = \left[\ln \left(\frac{a_{\max}}{a_{\min}} \right) \right]^{-1} \eta \left[\frac{T_2}{(1-\eta)} \ln \left(\frac{1 + e^{a_{\max}(1-\eta)/T_2}}{1 + e^{a_{\min}(1-\eta)/T_2}} \right) - T_1 \ln \left(\frac{1 + e^{a_{\max}/T_1}}{1 + e^{a_{\min}/T_1}} \right) \right]. \quad (49)$$

On the other hand, from the perspective of observer B who treats a_2 as the unknown parameter, the corresponding prior is $\pi(a_2) = [\ln(a_{\max}/a_{\min})]^{-1} (1/a_2)$. Upon writing the work per cycle as $W(a_2, \eta)$, i.e. function of a_2 and η , we define $\overline{W}(B) = \int W(a_2, \eta) \pi(a_2) da_2$, which is explicitly given by

$$\overline{W}(B) = \left[\ln \left(\frac{a_{\max}}{a_{\min}} \right) \right]^{-1} \eta \left[\frac{T_2}{(1-\eta)} \ln \left(\frac{1 + e^{a_{\max}/T_2}}{1 + e^{a_{\min}/T_2}} \right) - T_1 \ln \left(\frac{1 + e^{a_{\max}/(1-\eta)T_1}}{1 + e^{a_{\min}/(1-\eta)T_1}} \right) \right]. \quad (50)$$

In this case also, we see a difference in the average work expected by A and B, which is at variance with our assumption that A and B are in an equivalent state of knowledge. But again, we observe that in the asymptotic limit as considered in Section (V), both the expressions for work get reduced to the following simpler form

$$\overline{W}(A) \approx \overline{W}(B) \approx \frac{\ln 2}{\ln \left(\frac{a_{\max}}{a_{\min}} \right)} \eta \left(T_1 - \frac{T_2}{(1-\eta)} \right). \quad (51)$$

It is interesting to observe that the expected work in the asymptotic limit attains its optimal value at the well known Curzon-Ahlborn efficiency, $\eta = 1 - \sqrt{T_2/T_1}$. This value, near equilibrium is approximated as

$$\eta \approx \frac{(1-\theta)}{2} + \frac{(1-\theta)^2}{8} + O(1-\theta)^3. \quad (52)$$

Note the difference in the above dependence on θ , with the one estimated by Eq. (41) for the case of ignorance about two parameters.

Finally, we may compare the amount of work expected in the general case (as calculated from Eqs. (38) and (39)) when both a_1 and a_2 are uncertain, with the special case when the efficiency is fixed a priori. In the general case, the expected efficiency of the engine is given by Eq. (40), so it is reasonable to take this value of efficiency in Eq. (51) while making the above comparison. As shown in Fig. 3, the work in the general case is always less than the work in the special case and their ratio approaches the value of 3/4 as the temperature gradient goes to zero ($\theta \rightarrow 1$).

VII. CONCLUSIONS

The main issue of this paper is about predictability in models of quantum thermodynamic machines in the presence of incomplete knowledge about the working medium. The implicit reason for uncertainty considered here is different from other scenarios usually considered: The source of this uncertainty does not lie in some unknown intrinsic dynamics of the system or due to some environment induced fluctuations of the control parameters. We have not assumed any

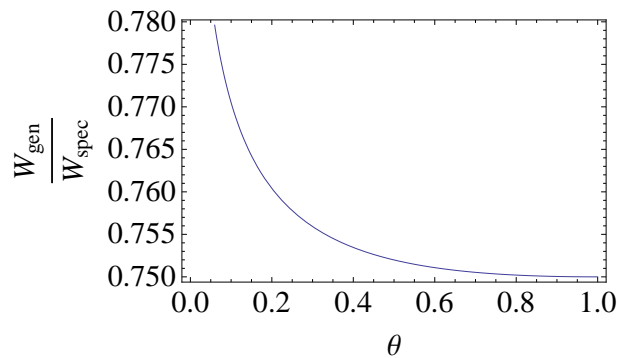


FIG. 3: Expected work in the general case, is less than the expected work in the special case for which the efficiency is fixed a priori. Both expressions for work are calculated at the same efficiency as given by Eq. (40). As $\theta \rightarrow 1$, the ratio approaches a constant value $3/4$.

such objective mechanism which may presumably cause these values to become uncertain. Also, the uncertainty does not stem from lack of information about the preparation procedure for the quantum state of the working medium.

We have assumed a Bayesian perspective about this (subjective) ignorance of the observer. In our model, there are two intrinsic energy scales of the working medium, which are under macroscopic control, for example, via external magnetic field. In our case, the hamiltonian itself is uncertain in as much as the values of control parameters entering in its description, are not pre-assigned.

Once we assume a Bayesian perspective on our prior ignorance about the system, then the task is to narrow down the choice of prior for the uncertain parameters. The prior so assigned is supposed to quantify the given prior information such as the knowledge that these parameters are positive valued and the criterion for operation as a heat engine. We make use of a consistency argument which says that different observers in possession of similar knowledge must assign similar priors. Invoking two observers A and B, an appropriate prior which we arrive at is the $1/x$ prior. Let us emphasize that in case further constraints or information from data become available, then the prior should be updated to include that additional information, by using suitable procedures [3] such as maximum entropy principle, or Bayes theorem. In the absence of data from observations, in our opinion, the initial prior so assigned has to be used to make inferences about physical quantities. The estimates for these quantities have been defined as the average value over the chosen prior.

Further considerations lead us to investigate a particular asymptotic limit, because to maintain consistency, the observers A and B should arrive at similar estimates for a given quantity, if each is in an equivalent state of knowledge. It is in this limit, we observe classical thermodynamic features for the estimated quantities of our quantum heat engine. In particular, the expected mean energies of the two-level systems become proportional to their temperature, with the expected heat capacity \overline{C} becoming independent of the temperature. It is also interesting to observe that the factor $\ln(a_{\text{max}}/a_{\text{min}})$ occurring in the normalisation of the prior can be expressed in terms of the expected heat capacity of the system. The estimated efficiency behaves like one-third of Carnot value, for close to equilibrium (nearly equal bath temperatures).

Further, if an additional constraint in the form of given value of efficiency is imposed, we observe that the number

of uncertain parameters in the problem reduces from two to one. It is then found that the expected work per cycle becomes optimal at CA efficiency. In Ref. [8], it was found that CA value is a lower bound for the efficiency at maximum work for this system. Proximity to this value at the optimal performance has been studied in many recent models and a certain universality has been observed [9, 10]. In this context, this predictability about the optimal behavior by using an appropriate prior for ignorance, is remarkable.

For the present model of heat engine, we have compared the estimated work per cycle in the general case of ignorance of two parameters with the special case of ignorance of one parameter only. Heuristically, one can anticipate that the expected work should increase when additional constraints are introduced which lead to reduced uncertainty about the system. In particular, the ratio of these two values of work, tends to a limiting value of $3/4$ as the two bath temperatures approach each other.

Similarly, the estimated efficiency close to equilibrium for the general case is one-third of Carnot value, as contrasted with the one-half Carnot value as found for the special case. This may indicate that the general case of prior ignorance about two parameters may belong to a different universality class than the special case of ignorance about a single parameter, which is bounded from below by half Carnot value [10, 11]. Further, we note that in a model of irreversible Brownian heat engine [12], when the power is optimised with respect to the load and the barrier height, the efficiency at optimal power is found to be given by

$$\eta^* = \frac{2(1 - \theta)^2}{3 - 2\theta(1 + \ln \theta) - \theta^2}. \quad (53)$$

Close to equilibrium, this efficiency has the same expansion as given by Eq. (41), upto second order.

Concluding, our analysis suggests an unexpected relation between the average performance of certain infinite time quantum models of heat engines inferred by a Bayesian analysis and the optimal performance of finite time thermodynamic models. In our opinion, the present study raises basic issues and open problems which we are addressing in future work, such as how robust is the predictability about efficiency at optimal work by choosing other models of engines, and to further investigate the influence of constraints on the expected behavior.

VIII. ACKNOWLEDGEMENT

RSJ acknowledges financial support from Department of Science and Technology, India under the project No. SR/S2/CMP-0047/2010(G).

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